The Largest Subsemilattices of the Semigroup of Transformations on a Finite Set

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Abstract

Let $T(X)$ be the semigroup of full transformations on a finite set X with n elements. We prove that every subsemilattice of $T(X)$ has at most 2^{n-1} elements and that there are precisely n subsemilattices of size exactly 2^{n-1}, each isomorphic to the semilattice of idempotents of the symmetric inverse semigroup on a set with $n-1$ elements.

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1 Introduction

A *semilattice* is a commutative semigroup consisting entirely of idempotents. That is, a semigroup S is a semilattice if and only if for all $a, b \in S$, $aa = a$ and $ab = ba$. A semilattice can also be defined as a partially ordered set $(S, \leq)$ such that the greatest lower bound $a \wedge b$ exists for all $a, b \in S$. Indeed, if S is a semilattice, then $(S, \leq)$, where $\leq$ is a relation on S defined by $a \leq b$ if $a = ab$, is a poset with $a \wedge b = ab$ for all $a, b \in S$. Conversely, if $(S, \leq)$ is a poset such that $a \wedge b$ exists for all $a, b \in S$, then S with multiplication $ab = a \wedge b$ is a semilattice. (See [3, page 10] and [7, Proposition 1.3.2].)

The class of semilattices forms a variety of algebras. In the lattice of the varieties of bands (idempotent semigroups), which has been described by Birjukov [3], Fennemore [5], and Gerhard [6], the variety of semilattices appears just above the trivial variety (as one of the three atoms) and below the varieties of left and right normal bands (see [2, Figure 1]). If S is an inverse semigroup (for every $a \in S$, there is a unique $a^{-1} \in S$ such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$), then the set $E(S)$ of idempotents of S is a semilattice. Therefore, semilattices play important role in the theory of inverse semigroups (see [7, Chapter 5]). They also appear in structure theorems for other classes of semigroups, for example for completely regular semigroups [7, Theorem 4.1.3]. In the theory of partially ordered sets, semilattices provide the most important generalization of lattices [3, page 22].

For a set X, we denote by $T(X)$ the semigroup of full transformations on X (functions from X to X), and by $I(X)$ the symmetric inverse semigroup of partial one-to-one transformations on X (one-to-one functions whose domain and image are included in X). In both cases, the operation is the composition of functions. (In this paper, we will write functions on the right: xf rather than $f(x)$, and compose from left to right: $x(fg) = (xf)g$ rather than $(fg)(x) = f(g(x))$.) For a semigroup S, denote by $E(S)$ the set of idempotents of S. The set $E(I(X))$ is a semilattice, consisting of all $e \in I(X)$ such that $\text{dom}(e) = \text{im}(e)$ and $xe = x$ for all $x \in \text{dom}(e)$. The semilattice $E(I(X))$,

viewed as a poset, is isomorphic to the poset $(\mathcal{P}(X), \subseteq)$ of the power set $\mathcal{P}(X)$ under inclusion. On the other hand, the set $E(T(X))$ is not a semigroup if $|X| \geq 3$ (if $|X| = 2$, then $E(T(X))$ is a semigroup but not a semilattice). It is therefore of interest to determine the subsets of $E(T(X))$ that are semilattices.

In 1991, Kunze and Crvenković [10] gave a criterion for a subsemilattice of $T(X)$ to be maximal, for a finite set X, in terms of transitivity orders on X introduced in [9]. Any subsemilattice S of $T(X)$ induces a partial order $\leq$ on X (transitivity order): $x \leq y$ if $x = y$ or $x = ye$ for some $e \in S$. Kunze and Crvenković have proved [10] that a subsemilattice S of $T(X)$ is a maximal subsemilattice if and only if the poset $(X \leq)$ satisfies certain conditions.

All these investigations have been prompted by the fact that any finite full transformation semigroup $T(X)$ is covered by its inverse subsemigroups [13] (also see [11, Theorem 6.2.4]), and by the still open problem, posed in [14], of describing the maximal inverse subsemigroups of $T(X)$. We note that if X is infinite, then $T(X)$ is not covered by its inverse subsemigroups [11, Exercise 6.2.8].

The purpose of this paper is to determine the largest subsemilattices of $T(X)$, where X is a finite set with n elements. We prove that for every subsemilattice S of $T(X)$, $|S| \leq 2^{n-1}$. Moreover, we exhibit the set of all subsemilattices of $T(X)$ with the maximum cardinality of 2^{n-1}. This set consists of n semilattices, each induced by one element of X and isomorphic to the semilattice $E(I(X'))$ of idempotents of the symmetric inverse semigroup $I(X')$, where X' is a set with $n-1$ elements.

In Section 2, we describe the idempotents of $T(X)$ and state some results about commuting idempotents. In Section 3, we define a collection $\{E_t\}_{t \in X}$ of n maximal subsemilattices of $T(X)$ with $|E_t| = 2^{n-1}$ for every $t \in X$. In the remainder of the paper, we prove that 2^{n-1} is the maximum cardinality that a subsemilattice of $T(X)$ can have, and that the semilattices E_t are the only subsemilattices of $T(X)$ that have this maximum cardinality. Our argument is by induction on n. The crucial step is a construction of a semilattice S^* (given a subsemilattice S of $T(X)$) such that $|S| \leq 2|S^*|$ and S^* can be embedded in $T(X')$, where $|X'| = n-1$. This construction is presented in Section 4. Finally, in Section 5, we state and prove our main results.

Throughout this paper, we fix a finite set X and reserve n to denote the cardinality of X. To simplify the language, we will say "semilattice in $T(X)$" to mean "subsemilattice of $T(X)$."

2 Idempotents in $T(X)$

For $a \in T(X)$, we denote by $\mathrm{im}(a)$ the image of a and by $\ker(a) = \{(x, y) \in X \times X : xa = ya\}$ the kernel of a. Let $e \in T(X)$ be an idempotent with the image $\{x_1, \ldots, x_k\}$. For each $i \in \{1, \ldots, k\}$, let $A_i = x_i e^{-1} = \{y \in X : ye = x_i\}$ and note that $A_i e = \{x_i\}$ and $x_i \in A_i$. The collection $\{A_1, \ldots, A_k\}$ is the partition of X induced by the kernel of e. We will use the following notation for e:

$$e = (A_1, x_1\rangle(A_2, x_2\rangle \ldots (A_k, x_k\rangle. \tag{2.1}$$

The following result has been obtained in [1] and [8].

Lemma 2.1. *Let $e = (A_1, x_1\rangle(A_2, x_2\rangle \ldots (A_k, x_k\rangle$ be an idempotent in $T(X)$ and let $a \in T(X)$. Then a commutes with e if and only if for every $i \in \{1, \ldots, k\}$, there is $j \in \{1, \ldots, k\}$ such that $x_i a = x_j$ and $A_i a \subseteq A_j$.*

Corollary 2.2. *Let e, f be idempotents in $T(X)$ such that $ef = fe$. Then, for all $x, y \in X$, if $x \in \mathrm{im}(e)$ and $yf \in xe^{-1}$, then $xf = x$.*

Lemma 2.3. *Let $x_1, \ldots, x_k$, where $k \geq 2$, be pairwise distinct elements of X. Let $e_1, \ldots, e_k$ be idempotents in $T(X)$ such that $x_i e_i = x_{i+1}$ for every i, $1 \leq i \leq k$, where we assume that $x_{k+1} = x_1$. Then there is $j \in \{2, \ldots, k\}$ such that $e_1 e_j \neq e_j e_1$.*

Proof. Suppose to the contrary that $e_1 e_j = e_j e_1$ for every $j \in \{2, \ldots, k\}$. Then $x_3 \in \mathrm{im}(e_2)$ and $x_2 e_1 = x_2 \in x_3 e_2^{-1}$, and so $x_3 e_1 = x_3$ by Corollary 2.2. Now, $x_4 \in \mathrm{im}(e_3)$ and $x_3 e_1 = x_3 \in x_4 e_3^{-1}$, and so $x_4 e_1 = x_4$ again by Corollary 2.2. Using the foregoing argument $k-1$ times, we obtain

that $x_{k+1}e_1 = x_{k+1}$, that is, $x_1e_1 = x_1$. But this is a contradiction since $x_1e_1 = x_2$. The result follows. $\square$

Lemma 2.3 implies the following proposition, which will be crucial for the construction presented in Section 4. The proposition is of an independent interest since, in part, it states that if S is a semilattice in $T(X)$, then the intersection of all images of the elements of S is not empty.

Proposition 2.4. *Let $|X| \geq 2$ and let S be a semilattice in $T(X)$. Then there are $t, u \in X$ with $u \neq t$ such that for every $e \in S$, $te = t$ and $ue \in \{u, t\}$.*

Proof. Suppose to the contrary that there is no $t \in X$ such that $te = t$ for every $e \in S$. Construct a sequence $y_1, y_2, y_3, \dots$ of elements of X and a sequence $f_1, f_2, f_3, \dots$ of elements of S as follows. Start with any $y_1 \in X$. By our assumption, there is $f_1 \in S$ such that $y_1f_1 \neq y_1$. Set $y_2 = y_1f_1$. By our assumption again, there is $f_2 \in S$ such that $y_2f_2 \neq y_2$. Set $y_3 = y_2f_2$, and continue the construction in the same way. Since X is finite, there is $p \geq 1$ such that $y_p = y_q$ for some $q > p$. Select the smallest such p, and then the smallest q such that $q > p$ and $y_p = y_q$. Note that $q > p+1$ since, by the construction, $y_{p+1} = y_pf_p \neq y_p$. By the minimality of p and q, $y_p, y_{p+1}, \dots, y_{q-1}$ are pairwise distinct, $y_if_i = y_{i+1}$ for every $i \in \{p, p+1, \dots, q-1\}$, and $x_{q-1}f_{q-1} = x_q = x_p$. Thus, by Lemma 2.3, there is $j \in \{p+1, \dots, q-1\}$ such that $f_pf_j \neq f_jf_p$, which is a contradiction. Hence a desired t exists.

Suppose to the contrary that there is no $u \in X$ such that $u \neq t$ and $ue \in \{u, t\}$ for every $e \in S$. Then we obtain a contradiction in the same way as in the first part of the proof. The only difference is that we start with $y_1 \neq t$ (which is possible since $|X| \geq 2$) and for each i, we select $f_i \in S$ such that $y_if \notin \{y_i, t\}$. Hence a desired u exists, which concludes the proof. $\square$

3 A Collection of Maximal Semilattices in $T(X)$

In this section, we define a collection $\{E_t\}_{t \in X}$ of n isomorphic maximal semilattices in $T(X)$ such that $|E_t| = 2^{n-1}$ for every $t \in X$. In Sections 4 and 5, we will prove that 2^{n-1} is the maximum cardinality of a semilattice in $T(X)$ and that $\{E_t\}_{t \in X}$ is the collection of all semilattices in $T(X)$ whose cardinality is 2^{n-1}.

Definition 3.1. Fix $t \in X$ and let $X_t = X - \{t\}$, so $|X_t| = n - 1$. For any $A = \{x_1, \dots, x_k\} \subseteq X_t$ (where $0 \leq k \leq n-1$, so A may be empty), define $e_A \in T(X)$ by

$$e_A = (X - A, t\rangle(\{x_1\}, x_1\rangle \dots (\{x_k\}, x_k\rangle$$

(see notation (2.1)). Note that $xe_A = x$ if $x \in A$, $xe_A = t$ if $x \notin A$, and $e_Ae_B = e_{A \cap B}$ for all $A, B \subseteq X_t$. We define a subset E_t of $T(X)$ by

$$E_t = \{e_A : A \subseteq X_t\}.$$

Recall that $I(X)$ denotes the symmetric inverse semigroup of all partial one-to-one transformations on X and that $E(I(X))$ is the semilattice of idempotents of $I(X)$. Every $\mu \in E(I(X))$ is completely determined by its domain: if $A = \mathrm{dom}(\mu)$, then $x\mu = x$ for every $x \in A$ (and $x\mu$ is undefined for every $x \notin A$). We will denote the idempotent in $I(X)$ with domain A by μ_A. Note that for all $A, B \subseteq X$, $\mu_A\mu_B = \mu_{A \cap B}$. The idempotent $\mu_\emptyset$ is the zero in $I(X)$. We agree that $I(\emptyset) = \{0\}$.

Proposition 3.2. *For every $t \in X$,*

(1) *E_t is a maximal semilattice in $T(X)$.*

(2) *E_t is isomorphic to $E(I(X_t))$ and $|E_t| = 2^{n-1}$.*

Proof. Let $t \in X$. For all $e_A, e_B \in E_t$, $e_A e_B = e_{A\cap B} \in E_t$, $e_A e_A = e_{A\cap A} = e_A$, and $e_A e_B = e_{A\cap B} = e_{B\cap A} = e_B e_A$. Thus E_t is a semilattice. Let $f \in T(X)$ be an idempotent such that $f \notin E_t$. By the definition of E_t, there are $y, z \in X$ such that $z \neq y$, $y \neq t$, and $zf = y$. Suppose $z = t$, so $tf = y$. Take $e_A \in E_t$ such that $y \notin A$. Then $t(fe_A) = ye_A = t$ and $t(e_A f) = tf = y$, so $fe_A \neq e_A f$. Suppose $z \neq t$. Take $e_A \in E_t$ such that $y \notin A$ and $z \in A$. Then $z(fe_A) = ye_A = t$ and $z(e_A f) = zf = y$, so $fe_A \neq e_A f$. It follows that E_t is a maximal semilattice in $T(X)$. We have proved (1).

Define $\phi : E_t \to E(I(X_t))$ by $e_A\phi = \mu_A$. Then clearly ϕ is a bijection and for all $e_A, e_B \in E_t$,

$$(e_A e_B)\phi = e_{A\cap B}\phi = \mu_{A\cap B} = \mu_A \mu_B = (e_A\phi)(e_B\phi).$$

Thus ϕ is an isomorphism. Finally, it is clear that the mapping $e_A \to A$ is a bijection from E_t onto $\mathcal{P}(X_t)$, and so $|E_t| = |\mathcal{P}(X_t)| = 2^{|X_t|} = 2^{n-1}$. We have proved (2). ◻

Nichols [12] has proved that for every $t \in X$, the set

$$I_t = \{a \in T(X) : ta = t \text{ and } |xa^{-1}| = 1 \text{ for all } x \in \mathrm{im}(a) - \{t\}\}$$

is a maximal inverse subsemigroup of $T(X)$. We note that E_t is the semilattice of idempotents of the inverse semigroup I_t.

4 An Inductive Construction

Throughout this section, we assume that $|X| \geq 2$ and we fix a semilattice S in $T(X)$ and $t, u \in X$ with $u \neq t$ such that for all $e \in S$, $te = t$ and $ue \in \{u, t\}$. (Such t and u exist by Proposition 2.4.) Our goal is to construct a semilattice S^* in $T(X)$ such that $|S| \leq 2|S^*|$ and S^* can be embedded in $T(X_u)$. (Recall that $X_u = X - \{u\}$, so $|X_u| = n - 1$.)

Definition 4.1. For $g \in S$, define $g^* \in T(X)$ by

$$xg^* = \begin{cases} xg & \text{if } xg \neq u, \\ t & \text{if } xg = u. \end{cases}$$

We define a subset S^* of $T(X)$ by $S^* = \{g^* : g \in S\}$.

Lemma 4.2. *S^* is a semilattice in $T(X)$.*

Proof. Define $\phi : S \to T(X)$ by $g\phi = g^*$. We claim that ϕ is a homomorphism. Let $g, h \in S$ and let $x \in X$. Suppose $x(gh) = u$. Then $x(gh)^* = t$ and $(xg)h^* = t$. If $xg \neq u$ then $xg^* = xg$, and so $x(g^*h^*) = (xg)h^* = t$. If $xg = u$ then $xg^* = t$, and so $x(g^*h^*) = th^* = th = t$. Hence $x(gh)^* = x(g^*h^*)$.

Suppose $x(gh) \neq u$. Then $x(gh)^* = x(gh)$ and $(xg)h^* = (xg)h$. If $xg \neq u$ then $xg^* = xg$, and so $x(g^*h^*) = (xg)h^* = (xg)h$. If $xg = u$ then $(xg)h = t$ (since $(xg)h = uh \in \{u, t\}$ and $(xg)h \neq u$), and so $x(g^*h^*) = th^* = th = t = (xg)h$. Hence $x(gh)^* = x(g^*h^*)$.

We have proved that $(gh)^* = g^*h^*$ for all $g, h \in S$, and the claim follows. It is clear that $\mathrm{im}(\phi) = S^*$. Hence S^* is a semilattice in $T(X)$ since any homomorphic image of a semilattice is a semilattice. ◻

Lemma 4.3. *Let $g, h \in S$ be such that $xg = u$ and $yh = u$ for some $x, y \in X$. Suppose $g \neq h$. Then $g^* \neq h^*$.*

Proof. Since $g \neq h$, there is $z \in X$ such that $zg \neq zh$. If $zg \neq u$ and $zh \neq u$, then $zg^* = zg \neq zh = zh^*$.

Suppose $zg = u$ or $zh = u$. We may assume that $zg = u$. Then $zg^* = t$. Since $zh \neq zg = u$, we have $zh^* = zh$. Since $gh = hg$,

$$(zh)g = (zg)h = uh = (yh)h = y(hh) = yh = u,$$

which implies that $zh \neq t$ (since $tg = t \neq u$). Thus $zg^* = t \neq zh = zh^*$.

Hence $zg^* \neq zh^*$ in all cases, and so $g^* \neq h^*$. ◻

Lemma 4.4. $|S| \le 2|S^*|$.

Proof. Let $A = \{g \in S : xg = u \text{ for some } x \in X\}$. Then, by the definition of S^*, we have $S^* = (S - A) \cup A^*$, where $A^* = \{g^* : g \in A\}$. Thus $S \subseteq S^* \cup A$, and so $|S| \le |S^* \cup A| \le |S^*| + |A|$. By Lemma 4.3, $|A^*| = |A|$, and so

$$|S| \le |S^*| + |A| = |S^*| + |A^*| \le |S^*| + |S^*| = 2|S^*|,$$

which concludes the proof. $\square$

We will now show that the semilattice S^* can be embedded in $T(X_u)$. For a function $f : A \to B$ and $A_0 \subseteq A$, we denote by $f|_{A_0}$ the restriction of f to A_0.

Definition 4.5. We define a subset S^*_u of $T(X_u)$ by

$$S^*_u = \{e \in T(X_u) : e = g^*|_{X_u} \text{ for some } g \in S\}.$$

Note that indeed $S^*_u \subseteq T(X_u)$ since $xg^* \ne u$ for all $g \in S$ and $x \in X$.

Lemma 4.6. *S^*_u is a semilattice in $T(X_u)$ isomorphic to S^*.*

Proof. Define $\phi : S^* \to T(X_u)$ by $g^*\phi = g^*|_{X_u}$. Then ϕ is a homomorphism since for all $g^*, h^* \in S^*$,

$$(g^*h^*)\phi = (g^*h^*)|_{X_u} = (g^*|_{X_u})(h^*|_{X_u}) = (g^*\phi)(h^*\phi).$$

Clearly $\mathrm{im}(\phi) = S^*_u$. Suppose $g^*|_{X_u} = h^*|_{X_u}$. Then $xg^* = xh^*$ for every $x \in X_u$. Thus $g^* = h^*$ since $ug^* = t = uh^*$. Hence ϕ is one-to-one, and the result follows. $\square$

5 The Maximum Cardinality Results

In this section, we prove our main results about the maximum cardinalities of semilattices in $T(X)$.

Lemma 5.1. *Let $|X| \ge 2$, S be a semilattice in $T(X)$, and $t, u \in X$ be such that $u \ne t$ and for every $e \in S$, $te = t$ and $ue \in \{u, t\}$. Suppose that $|xe^{-1}| \le 1$ for all $e \in S$ and all $x \in X - \{u, t\}$, and that $|uf^{-1}| \ge 2$ for some $f \in S$. Then $|S| < 2^{n-1}$.*

Proof. Define $\lambda : S \to E_t$ by

$$x(e\lambda) = \begin{cases} xe & \text{if } x = u, \\ xe & \text{if } x \ne u \text{ and } xe \ne u, \\ t & \text{if } x \ne u \text{ and } xe = u, \end{cases}$$

where $e \in S$ and $x \in X$. Note that, indeed, $e\lambda \in E_t$ since $|xe^{-1}| \le 1$ if $x \notin \{t, u\}$, so $|x(e\lambda)^{-1}| \ge 1$ if $x \ne t$. We claim that λ is one-to-one but not onto.

Let $g, h \in S$ be such that $g\lambda = h\lambda$ and let $x \in X$. If $x = u$, then $xg = x(g\lambda) = x(h\lambda) = x\lambda$. If $x \ne u$, $xg \ne u$, and $xh \ne u$, then again $xg = x(g\lambda) = x(h\lambda) = x\lambda$.

Let $x \ne u$. Suppose $xg = u$ or $xh = u$. We may assume that $xg = u$. Then $x(g\lambda) = t$, and so $x(h\lambda) = x(g\lambda) = t$. The latter implies that either $xh = t$ or $xh = u$. We claim that $xh = t$ is impossible. Indeed, suppose to the contrary that $xh = t$. Then, since $gh = hg$,

$$t = tg = (xh)g = (xg)h = uh = u(h\lambda) = u(g\lambda) = ug = (xg)g = x(gg) = xg = u,$$

which is a contradiction since $t \ne u$. Thus $xh \ne t$, and so $xh = u$. But then $xg = xh$.

Hence $xg = xh$ for every $x \in X$, and so λ is one-to-one. We know that there exists $f \in S$ such that $|uf^{-1}| \ge 2$. Thus there is $z \in X$ such that $z \ne u$ and $zf = u$. Consider any $e_A \in E_t$ such that $z \in A$ and $u \notin A$. Note that $ze_A = z$ and $ue_A = t$. We claim that $e_A \notin \mathrm{im}(\lambda)$. Suppose to the contrary

that $e_A = g\lambda$ for some $g \in S$. Then there is no $x \in X$ such that $xg = u$. (Indeed, if such an x existed, then we would have $ug = (xg)g = x(gg) = xg = u$, and so $t = ue_A = u(g\lambda) = ug = u$, which would contradict $u \neq t$.) It then follows from the definition of λ that $g\lambda = g$. Hence $e_A = g\lambda = g \in S$, and so $e_A f = f e_A$. But $z(fe_A) = ue_A = t$ and $z(e_A f) = zf = u$, and we have obtained a contradiction since $u \neq t$. The claim has been proved, and so λ is not onto.

Since $\lambda : S \to E_t$ is one-to-one but not onto, we have $|S| < |E_t| = 2^{n-1}$. $\square$

We can now prove our main results.

Theorem 5.2. *Let S be a semilattice in $T(X)$, where X is a finite set with n elements. Suppose that $S \neq E_s$ for every $s \in X$. Then $|S| < 2^{n-1}$.*

Proof. We proceed by induction on n. The result is vacuously true for $n = 1$. Let $n \geq 2$ and suppose the result is true for every semilattice in $T(Z)$ with $|Z| = n - 1$. By Proposition 2.4, there are $t, u \in X$ such that $u \neq t$ and for every $e \in S$, $te = t$ and $ue \in \{u, t\}$. We consider two cases.

Case 1. $|xe^{-1}| \leq 1$ for all $e \in S$ and all $x \in X - \{u, t\}$.

Then, since $S \neq E_t$, there is $f \in S$ such that $|uf^{-1}| \geq 2$, and so $|S| < 2^{n-1}$ by Lemma 5.1.

Case 2. $|yg^{-1}| \geq 2$ for some $g \in S$ and some $y \in X - \{u, t\}$.

Then there is $z \in X$ such that $z \neq y$ and $zg = y$. Note that $z \neq u$ (since $ug \in \{u, t\}$ and $y \neq u, t$). Consider the semilattice S^*_u in $T(X_u)$ from Definition 4.5. For any $s \in X_u$, we now have the semilattice E_s in $T(X)$ and the semilattice E_s in $T(X_u)$. To avoid ambiguity, we will denote the latter by E'_s.

Consider $e = g^*|_{X_u} \in S^*_u$. By the definition of g^* (see Definition 4.1), $ze = zg^* = zg = y$, and so $e \notin E'_t$ (since $y \neq t$ and $z \neq y$). Thus $S^*_u \neq E'_t$, and so $S^*_u \neq E'_s$ for every $s \in X_u$ (since $te = t$ for every $e \in S^*_u$). Hence, by the inductive hypothesis, $|S^*_u| < 2^{n-2}$. But $|S^*_u| = |S^*|$ (by Lemma 4.6) and $|S| \leq 2|S^*|$ (by Lemma 4.4). Hence $|S| \leq 2|S^*| = 2|S^*_u| < 2 \cdot 2^{n-2} = 2^{n-1}$. $\square$

The following corollary follows immediately from Theorem 5.2 and Proposition 3.2.

Corollary 5.3. *Let X be a finite set with n elements. Then:*

(1) *The maximum cardinality of a semilattice in $T(X)$ is 2^{n-1}.*

(2) *There are exactly n semilattices in $T(X)$ of cardinality 2^{n-1}, namely the semilattices E_t ($t \in X$) from* Definition 3.1.

(3) *Each semilattice E_t is isomorphic to the semilattice of idempotents of the symmetric inverse semigroup $I(X')$, where $|X'| = n - 1$.*

By Corollary 5.3, if S is a semilattice in $T(X)$ with $m = |S|$, then $1 \leq m \leq 2^{n-1}$. It is easy to see that the converse is also true: if $1 \leq m \leq 2^{n-1}$, then $m = |S|$ for some semilattice S in $T(X)$. Indeed, fix $t \in X$. If $m = 2^{n-1}$, then $m = |E_t|$. Let $1 < m \leq 2^{n-1}$ and suppose $m = |S|$ for some subsemilattice S of E_t. Let $A \subseteq X_t$ be such that $e_A \in S$ and $|A| \geq |B|$ for every $B \subseteq X_t$. Then $S_1 = S - \{e_A\}$ is a subsemilattice of E_t with $|S_1| = m - 1$. The converse follows.

Of course, the subsemilattices of E_t, with the exception of E_t itself, are not maximal. We conclude the paper with the following problem.

Problem. Let X be a finite set with n elements. Which numbers m, $1 \leq m \leq 2^{n-1}$, can serve as the cardinalities of maximal semilattices in $T(X)$?

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